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From Q-walls to Q-balls

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Abstract: We study Q-ball type solitons in arbitrary spatial dimensions in the setting recently described by Kusenko, where the scalar field potential has a flat direction which rises much slower than ϕ^2. We find that the general formula for energy as a function of the charge is, $E_d \sim Q_d^{(d/d+1)}$ in spatial dimension d. We show that the hamiltonian governing the stability analysis of certain Q-wall configurations, which are one-dimensional Q-ball solutions extended to planar, wall-like configurations in three dimensions, can be related to supersymmetric quantum mechanics. Q-wall and Q-string (the corresponding extensions of two-dimensional Q-balls in 3 spatial dimensions) configurations are seen to be unstable, and will tend to bead and form planar or linear arrays of three-dimensional Q-balls. The lifetime of these wall-like and string-like configurations is, however, arbitrarily large and hence they could be of relevance to cosmological density fluctuations and structure formation in the early Universe.

Keywords: Solitons Monopoles and Instantons, Global Symmetries, Physics of the Early Universe.

Dedicated to our loves, our lives, . . .
our wives

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Contents

1. Introduction

Non-topological solitons were first described in the mid-70's [1] in the context of scalar field models with two fields, symmetry breaking and a conserved charge. They reappeared in a somewhat different guise in the mid 80's in theories without spontaneous symmetry breaking as considered by Coleman [2] and the name Q-balls was invented. In the mid 90's they have re-surfaced in connection with models of scalar fields where the potential has flat or quasi-flat directions, which is generic in models of supersymmetry breaking, as described by Kusenko [3].

The original non-topological solitons and Q-balls had an energy which behaved as

$$E \sim \mu Q \tag{1.1}$$

for large conserved charge Q. If μ is less than the mass of a perturbative, charged excitation of unit charge, then the Q-balls are clearly stable against decay into perturbative charged particles. Indeed, Coleman [2] defined Q balls as the minimal energy classical configurations of fixed charge Q and was able to show their absolute, classical stability. However, it was soon realized that linear terms induced via quantum corrections in the effective potential would lead to the decay of these Q-balls [4].

The recently-described Q-balls do not suffer this fate. They have an intriguing energy-charge dispersion, $E \sim Q^{(3/4)}$ for large Q in 3 spatial dimensions. Hence these Q-balls are arbitrarily tightly bound compared to the energy of an aggregate of perturbative quanta of equal charge. We note that such an asymptotic energy-charge dispersion with the non-integer power 3/4 also arises in another context. For solitons in models with non-trivial topological Hopf windings [5], there is an energy bound, which is not attained, but behaves as $Q^{(3/4)}$. It would be interesting to find a model where a "Hopf soliton", perhaps of the Q-ball, type can exactly saturate such a bound. Indeed, we will find below exactly solvable Q-balls which do saturate a similar bound in 1 spatial dimension.

In this paper we investigate extended objects of the Q-ball type. These are the solitons of one- and two-dimensional models embedded in three dimensions, forming Q-walls and Q-strings, respectively. Similar structures have been studied recently in [10, 11] both analytically and using numerical simulations. The cases discussed concern Q-balls which have an energy-charge dispersion that grows linearly with the charge, $E \sim \mu Q$. It is found that collisions of such Q-balls can give rise to Q-string like states. In [11] stable Q-ring solitons are described. Here stability is achieved via an additional topological twisting of the phase as one goes around the ring. We find our study is complementary to these studies.

In the next section, we study Q-solitons in arbitrary spatial dimension d. The energetics of these objects are interesting: the energy E_d of a soliton in d dimensions depends on its charge Q_d as $E_d \sim Q_d^{d/(d+1)}$. The lower the dimension, the more energetically favourable the Q-soliton over a conventional configuration of charge Q_d. At first glance, this would seem to indicate that extended objects in three dimension would be energetically preferred over Q-balls. However, as we shall see, this turns out not to be the case.

In section 3, we consider embedded lower-dimensional Q-solitons in three dimensions. The energy E_2 and charge Q_2 of a two-dimensional soliton, for instance, are the energy and charge per unit length of the corresponding Q-string. The energy of a Q-string of length L and of fixed three-dimensional charge varies as $L^{1/3}$. Thus the tension varies as $L^{-2/3}$, reducing rapidly with length. However, the energy per unit length also varies as $L^{-2/3}$, so the natural time scale for oscillations of Q-strings is equivalent to that of topological strings (whose energy per unit length and tension are independent of L).

In section 4 we study the stability of Q-strings or walls against "beading", i.e. the formation along the length of the string (or surface of the wall) of clumps of higher charge (and energy) per unit length (or surface) which would ultimately result in the formation of a one-dimensional chain (or two-dimensional sheet) of non-interacting balls.

We study the stability of walls, in particular, since there the analysis is much simpler, essentially because the one-dimensional soliton can be found analytically for certain potentials. In studying small oscillations about such a solution, the eigenvalue equation determining the frequency of these oscillations can be related to a Schrödinger equation in supersymmetric quantum mechanics. Applying known results from that theatre enables us to identify a single unstable mode for the fluctuation, two zero-modes (one of which corresponds to translation of the wall) and a spectrum of stable modes. The unstable mode can have an arbitrarily large characteristic time, so the wall can be extremely long-lived. We expect the same behaviour for strings, but have not yet done the analysis in that case.

2. Q-balls in any dimension

Consider a charged scalar field in d spatial dimensions, the lagrangian governing its behaviour is

$$\mathcal{L} = \int d^dx \left(\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - V(|\phi|) \right), \tag{2.1}$$

where $a = 1, 2$, $|\phi| = \sqrt{\phi_1^2 + \phi_2^2}$ and we use a simple SO(2) global gauge invariance whose conserved charge corresponds to Q. The conserved charge is normalized as

$$Q = \int d^dx \, \epsilon_{ab} \phi_a \partial_t \phi_b \,. \tag{2.2}$$

The scalar potential, inspired by supersymmetric models, satisfies

$$V(|\phi|) = \begin{cases} \frac{1}{2} m^2 |\phi|^2 \,, & |\phi| \to 0 \,, \\ \Lambda \,, & |\phi| \to \infty \,, \end{cases} \tag{2.3}$$

where Λ is a constant which is determined by the scale of supersymmetry breaking. The minimum of the potential occurs at $|\phi| = 0$ and the charge symmetry is unbroken. Q-ball solutions are found with the ansatz

$$\phi_1 + i\phi_2 = e^{i\omega t} \varphi(r) \,, \tag{2.4}$$

where ω is a constant, r is the radial coordinate and $\varphi(r)$ is real. We immediately obtain

$$Q = \omega \int d^dx \, \varphi^2(r) \,. \tag{2.5}$$

The profile taken for $\varphi(r)$ satisfies the asymptotic conditions

$$\varphi(r) = \begin{cases} \phi_0\,, & r \approx 0\,, \\ 0\,, & r \to \infty\,, \end{cases} \tag{2.6}$$

where ϕ_0 is taken large enough so that the asymptotic behaviour of the potential is valid. Hence the Q-ball interpolates from large non-vacuum values around ϕ_0 in its center to $\varphi = 0$, the perturbative vacuum, at infinity. The existence of stable solutions to the equations of motion for $\varphi(r)$ in 3 spatial dimensions is the content of the previous papers [2, 6] and is easiest seen by the classic continuity arguments of Coleman [2]. Here we re-do the analysis in arbitrary spatial dimensions d.

The equations of motion for $\varphi(r)$ are

$$\nabla^2\varphi(r) - V'(\varphi(r)) + \omega^2\varphi(r) = 0 \tag{2.7}$$

which, given the form of the potential, become

$$\begin{aligned} \nabla^2\varphi(r) + \omega^2\varphi(r) &= 0\,, \qquad r \approx 0\,, \\ \nabla^2\varphi(r) - m^2\varphi(r) &= 0\,, \qquad r \approx \infty \end{aligned} \tag{2.8}$$

using the assumed asymptotics of the potential. The solutions of these equations are immediate, but dimension dependent. For small r we get spherical Bessel functions of order 0 in 3 dimensions, ordinary Bessel functions of order 0 in 2 dimensions, and trigonometric functions in 1 dimension, with argument ωr. For large r the argument simply changes to imr.

2.1 $d = 3$

The asymptotic solutions are

$$\varphi(r) = \begin{cases} \phi_0 j_0(\omega r)\,, & r \approx 0\,, \\ \varphi_0 k_0(mr)\,, & r \approx \infty\,, \end{cases} \tag{2.9}$$

where ϕ_0 is a constant that is fixed by the total charge of the solution, φ_0 is a dependent constant and $j_0(\omega r)$ and $k_0(mr)$ are the spherical Bessel function of order 0 of real and imaginary argument, respectively. The asymptotic solutions interpolate from one to the other in a relatively small region, and it is difficult to obtain analytic solutions, except in the one-dimensional case. However, the continuity argument of Coleman easily establishes the existence of a solution.

The idea is to first observe that the equation of motion (2.7) can be interpreted to describe the classical motion of a "particle" in one dimension of position φ, in the potential $-V(\varphi) + \frac{1}{2}\omega^2\varphi^2$ and in the presence of a specific frictional term that is time dependent coming from the laplacian, and where r plays the role of time. Indeed the equation of motion is equivalently written as

$$\ddot{\varphi}(r) - V'(\varphi(r)) + \omega^2\varphi(r) = -\frac{2}{r}\dot{\varphi}(r)\,, \tag{2.10}$$

where $\dot{\varphi}(r) = \partial_r\varphi(r)$. The potential is negative for small φ, and is approximately a downward opening parabola of curvature $-m^2$, whose maximum passes through the origin at $\varphi(r) = 0$. It turns over, after achieving a minimum (negative) value, into an upward opening parabola of curvature ω^2 for large enough values of $\varphi(r)$. Clearly, eventually the potential becomes positive. Now if we start with an initial value ϕ_0 for which $V(\phi_0)$ is negative, the "particle" will never rise up high enough to reach the origin, even in infinite time, and will perform damped oscillations till it settles down to the global minimum value of $-V(\varphi) + \frac{1}{2}\omega^2\varphi^2$. On the other hand if we start with a large initial value of ϕ_0, then since the "motion" is governed arbitrarily well by the spherical Bessel function $j_0(\omega r)$, the "particle" will attain a given value close to the point where the full potential $-V(\varphi) + \frac{1}{2}\omega^2\varphi^2$ vanishes, in a fixed time, independent of the initial value ϕ_0 which is of the order of $1/\omega$. The velocity at this point will be proportional to ϕ_0. Hence the subsequent evolution can be arranged to commence with an arbitrarily high initial velocity by choosing ϕ_0 larger and larger. It is clear then, that there exists an initial value for ϕ_0 so that the "particle" will arrive at this point with sufficient velocity that the subsequent motion will rise up and go over the maximum at $\varphi = 0$ in finite time. Hence, by continuity, there exists an initial value for ϕ_0 which will exactly achieve the maximum as time goes to infinity, i.e. for $r \to \infty$. Such a configuration is the solution and it will give the appropriate interpolation between the two asymptotic solutions (2.9). Clearly the value of φ_0 will be governed by the value of ϕ_0 which in turn will be fixed by the value ω. Hence the total charge Q can be thought of as a function of ω alone, or conversely all of the parameters depend only on the total charge Q.

The energy of such a configuration can be approximated as

$$\begin{aligned} E &\approx \int_{r<1/\omega} d^3x\ \left(\omega^2\varphi^2(r) + |\nabla\varphi(r)|^2 + \Lambda\right) = \int_{r<1/\omega} d^3x\ \left(2\omega^2\varphi^2(r) + \Lambda\right) \\ &= \frac{\alpha\phi_0^2}{\omega} + \frac{\beta\Lambda}{\omega^3}, \end{aligned} \tag{2.11}$$

where we neglect the contributions from the surface and the exterior of the Q-ball, and where α and β are positive constants. This expression is of course reasonable only for Q-balls with large charge Q. Using

$$Q = \omega\int d^3\, x\varphi^2(r) \approx \omega\int_{r<1/\omega} d^3x\ \phi_0^2 j_0^2(\omega r) = \frac{\gamma\phi_0^2}{\omega^2}, \tag{2.12}$$

where γ is another positive constant, we get

$$E \approx \frac{\alpha Q\omega}{\gamma} + \frac{\Lambda\beta}{\omega^3}. \tag{2.13}$$

Imposing $dE/d\omega = 0$ yields

$$\frac{\alpha Q}{\gamma} = \frac{3\Lambda\beta}{\omega^4} \tag{2.14}$$

which gives

$$\begin{aligned}\omega &\sim Q^{-1/4}\,,\\ \phi_0 &\sim Q^{1/4}\,,\\ E &\sim Q^{3/4}\,.\end{aligned} \tag{2.15}$$

2.2 $d = 2$

Here the solution for small r is $\phi_0 J_0(\omega r)$ where $J_0(\omega r)$ is the ordinary Bessel function of order 0, and for large r the solution is $\varphi_0 K_0(mr)$, where $K_0(mr)$ is the Bessel function of order zero of imaginary argument. The energy is then approximately given by

$$\begin{aligned}E_2 &\approx \int_{r<1/\omega} d^2x\,\left(\omega^2\varphi^2(r) + |\nabla\varphi(r)|^2 + \Lambda\right)\\ &= \frac{\alpha Q_2 \omega}{\gamma} + \frac{\beta\Lambda}{\omega^2}\end{aligned} \tag{2.16}$$

which yields upon imposing $dE_2/d\omega = 0$

$$\begin{aligned}\omega &\sim Q_2^{-1/3}\,,\\ \phi_0 &\sim Q_2^{1/3}\,,\\ E_2 &\sim Q_2^{2/3}\,.\end{aligned} \tag{2.17}$$

Here E_2 and Q_2 are the appropriate two-dimensional energy and charge, respectively, which in terms of three-dimensional energy and charge are the energy per unit length and the charge per unit length.

2.3 $d = 1$

Here the equation of motion can be reduced to quadratures, and it can be integrated exactly in many cases. It is more convenient to use the normal spatial coordinate $x \in [-\infty, \infty]$ rather than the radial coordinate r. The equation of motion for $\varphi(x)$ where $\phi_1(x) + i\phi_2(x) = e^{i\omega t}\varphi(x)$ is

$$\frac{d^2\varphi(x)}{dx^2} - V'(\varphi(x)) + \omega^2\varphi(x) = 0 \tag{2.18}$$

which yields

$$\int_{\phi_0}^{\varphi(x)} \frac{d\varphi}{\pm\sqrt{V(\varphi) - \omega^2\varphi^2}} = \int_{x_0}^{x} dx\,. \tag{2.19}$$

Here x_0 gives the position of the Q-ball and the $\pm$ is chosen depending on whether $x > x_0$ or $x < x_0$. The square root should not become imaginary thus we must have that $V(\varphi) > \omega^2\varphi^2$. Since for large values of φ the potential becomes flat and the required inequality is violated, there is a fixed maximum value for ϕ_0 given by the condition

$V(\phi_0) = \omega^2\phi_0^2$. There is no "friction" term coming from the laplacian in 1 dimension, hence conservation of energy dictates that this value of ϕ_0 is the starting point for $\varphi_0(x)$. Small ω implies large values of ϕ_0 which gives approximately that $\phi_0^2 \approx \Lambda/\omega^2$ that is $\phi_0 \sim 1/\omega$. The quadrature (2.19) is not particularly useful in the case of a general potential $V(\varphi(x))$. The solution for $x \approx x_0$ is $\varphi(x) = \phi_0 \cos(\omega(x - x_0))$ while for $|x - x_0| \gg \pi/2\omega$ is $\varphi(\pm x) = \varphi_0 e^{\mp mx}$. The charge is given by

$$Q_1 = \omega \int dx\varphi^2(x) \approx \omega\phi_0^2 \int_{x_0-(\pi/2\omega)}^{x_0+(\pi/2\omega)} dx\ \cos^2(\omega(x - x_0)) = \frac{\phi_0^2\pi}{2} \tag{2.20}$$

while the energy is

$$\begin{aligned}
E_1 &= \int dx\ \big(\omega^2\varphi^2(x) + (\partial_x\varphi(x))^2 + V(\varphi(x))\big) \\
&= \int dx\ \big(2\omega^2\varphi^2 + V(\varphi(x)) - \varphi(x)V'(\varphi(x))\big) \\
&\approx \int_{|x-x_0|<(\pi/2\omega)} dx\ \big(2\omega^2\varphi^2 + \Lambda\big) \\
&= 2\omega^2\phi_0^2 \left(\int_{|x-x_0|<(\pi/2\omega)} dx \cos^2(\omega(x - x_0))\right) + \frac{\Lambda\pi}{\omega} \\
&= \frac{2\pi\omega\phi_0^2}{2} + \frac{\Lambda\pi}{\omega}\,.
\end{aligned} \tag{2.21}$$

Imposing

$$\frac{dE_1}{d\omega} = 0 \tag{2.22}$$

yields

$$\phi_0^2 - \frac{\Lambda}{\omega^2} = \frac{2Q_1}{\pi} - \frac{\Lambda}{\omega^2} = 0 \tag{2.23}$$

implying

$$\omega^2 = \frac{\pi\Lambda}{2Q_1}\,. \tag{2.24}$$

This gives,

$$\begin{aligned}
\omega &\sim Q_1^{-1/2}\,, \\
\phi_0 &\sim Q_1^{1/2}\,, \\
E_2 &\sim Q_1^{1/2}\,.
\end{aligned} \tag{2.25}$$

2.4 Arbitrary spatial dimension

Here we record the corresponding formulae in arbitrary spatial dimension d. The Q_d-ball solution will be to the appropriate generalization of the free spherical wave in its centre with argument ωr while from its edge it will evolve into the exponentially decaying type solution of mass m outside, i.e. the same function but of imaginary argument imr. One easily finds

$$\begin{aligned}Q_d \approx \frac{\gamma_d \omega \phi_0^2}{\omega^d} &= \frac{\gamma_d \phi_0^2}{\omega^{(d-1)}} E_d \approx \alpha_d \phi_0^2 \omega^{(2-d)} + \frac{\beta_d \Lambda}{\omega^d} \\ &= \frac{\alpha_d Q_d \omega^{(d-1)}}{\gamma_d} \omega^{(2-d)} + \frac{\beta_d \Lambda}{\omega^d} \\ &= \frac{\alpha_d Q_d \omega}{\gamma_d} + \frac{\beta_d \Lambda}{\omega^d}\,, \end{aligned} \tag{2.26}$$

where α_d, β_d and γ_d are the corresponding constants in d dimensions. Imposing $dE_d/d\omega = 0$ yields in a straightforward manner

$$\begin{aligned}\omega &\sim Q_d^{-1/(d+1)}\,, \\ \phi_0 &\sim Q_d^{1/(d+1)}\,, \\ E_d &\sim Q_d^{d/(d+1)}\,. \end{aligned} \tag{2.27}$$

3. Energetics of Q-strings and Q-walls

Suppose initial conditions have created a large Q-wall or Q-string in 3 dimensions. We will observe here that such structures have interesting energetics. We address the question of stability of such objects in the next section.

3.1 Q-strings

We define Q-strings as linear configurations in 3 dimensions whose cross section corresponds to a two-dimensional Q_2-ball, with a fixed charge per unit length Q_2. Such a configuration will simply terminate at two end points or close on itself yielding a closed Q-string configuration. We will consider for convenience the second situation, of a closed Q-string of length L. We will assume that L is sufficiently long so that gradient energies due to the gradual bending that is necessary to close the Q-string are negligible. The relevant conserved charge of a closed Q-string then is given by

$$Q = Q_2 L \tag{3.1}$$

while the energy is given by

$$E = E_2 L\,. \tag{3.2}$$

Replacing for $E_2 \sim Q_2^{2/3} = (Q/L)^{2/3}$ we get

$$E \sim Q_2^{2/3} L = Q^{2/3} L^{1/3}\,. \tag{3.3}$$

Hence a Q-string of total charge Q satisfies an energy-charge dispersion relation

$$E_{Q\text{-string}} \sim Q^{2/3}\,. \tag{3.4}$$

This does not mean that the Q-string is even more tightly bound than a Q-ball with the same charge, which satisfies $E \sim Q^{3/4}$. Overall scaling of the Q-string always lowers its energy. The energy-length dispersion is

$$E_{Q\text{-string}} \sim L^{1/3} \tag{3.5}$$

so that as the length decreases, so does the energy. Indeed the Q-string will shrink and reduce its energy until its length is comparable to its width, at which point the description in terms of a Q-string is no longer valid.

The energy-length dispersion is however, remarkable. The restoring force which tends to cause the Q-string to shrink

$$F_{Q\text{-string}} = -\frac{dE}{dL} \sim -\frac{1}{3}\frac{Q^{2/3}}{L^{2/3}} \tag{3.6}$$

disappears as L gets very large. This is in contradistinction to the case of a topological string, such as a cosmic string or vortex loop, where the energy-length dispersion is $E_{\text{top.-string}} = \mu L$, where μ is the mass per unit length, giving a constant restoring force. Nevertheless, the time scale for shrinking of a Q-string is not markedly different, for a large Q-string the mass per unit length also decreases accordingly with its length, compensating for the reduced restoring force.

The time scale for shrinking of a topological string is governed by the effective lagrangian

$$\mathcal{L}_{\text{top.-string}} = \frac{1}{2}\mu L \frac{\dot{L}^2}{4\pi^2} - \mu L = \mu \left(\frac{L\dot{L}^2}{8\pi^2} - L \right) \tag{3.7}$$

while for a Q-string we have

$$\mathcal{L}_{Q\text{-string}} = \frac{1}{2}\left(\frac{\alpha Q_2^{2/3}}{L^{2/3}} \right) L \frac{\dot{L}^2}{4\pi^2} - \left(\frac{\alpha Q_2^{2/3}}{L^{2/3}} \right) L = \alpha Q_2^{2/3} \left(\frac{L^{1/3}\dot{L}^2}{8\pi^2} - L^{1/3} \right). \tag{3.8}$$

We see that neither μ nor $\alpha Q_2^{2/3}$ play a role in the dynamics. Conservation of energy for the topological string gives

$$\frac{L\dot{L}^2}{8\pi^2} + L = L_0 \tag{3.9}$$

while for the Q-string

$$\frac{L^{1/3}\dot{L}^2}{8\pi^2} - L^{1/3} = L_0^{1/3}\,, \tag{3.10}$$

where L_0 is the initial length in each case. With the change of variables $L = v^2$ for the topological string and $L = u^6$ for the Q-string the equations become easily integrable, yielding the time for contraction $T_{\text{top.-string}} = L_0/4\sqrt{2}$ and $T_{Q\text{-string}} = 45L_0/32\sqrt{2}$.

JHEP08(2001)003

We will see in the next section that the Q-string and the Q-wall are both unstable to beading into arrays of smaller Q-balls. If the Q-string were to spontaneously decay into a linear array of N, Q-balls each of charge q, we have

$$Q = Nq \tag{3.11}$$

from conservation of charge, while the energy of such a final state is

$$E = \beta q^{3/4} N = \beta Q^{3/4} N^{1/4} . \tag{3.12}$$

Evidently such a state has higher energy than the absolute minimum of energy in the sector of charge Q, given by the Q-ball of charge Q, ($N = 1$), which has energy $E = \beta Q^{3/4}$. But it will be an intermediate state with less energy than the Q-string if (adopting the notation of section 2.4 to avoid confusion)

$$\beta_3 Q^{3/4} N^{1/4} < \beta_2 Q^{2/3} L^{1/3} . \tag{3.13}$$

This implies

$$N < \left(\frac{\beta_2}{\beta_3}\right)^4 \frac{L^{4/3}}{Q^{1/3}} \tag{3.14}$$

which gives $N/L < (\beta_2/\beta_3)^4 Q_2^{-1/3}$. As the size of each smaller Q-ball is approximately $R \sim L/N$ thus $R > (\beta_3/\beta_2)^4 Q_2^{1/3}$, i.e. the Q-string will be stable against decay into a chain of N, Q-balls if the size of each Q-ball is smaller than $Q_2^{1/3} \sim 1/\omega$, which is just the width of the original Q-string.

3.2 Q-walls

We can repeat the above analysis for the case of a closed Q-wall of total three-dimensional charge Q and with a surface area L^2. Then

$$Q = L^2 Q_1 \tag{3.15}$$

and

$$E_{Q\text{-wall}} = L^2 E_1 = L^2 \beta_1 Q_1^{1/2} = \beta_1 L^2 \left(\frac{Q}{L^2}\right)^{(1/2)} = \beta_1 Q^{(1/2)} L . \tag{3.16}$$

This is to be compared with the energy of a topological domain wall,

$$E_{\text{top.-wall}} = \mu L^2 , \tag{3.17}$$

where μ is the mass per unit area. For a spherical surface the effective lagrangian describing the dynamics of a spherical topological domain wall is

$$\mathcal{L}_{\text{top.-wall}} = \frac{1}{2}\mu L^2 \frac{\dot{L}^2}{4\pi} - \mu L^2 \tag{3.18}$$

while for a spherical Q-wall we get

$$\mathcal{L}_{Q\text{-wall}} = \frac{1}{2}\beta_1 Q^{1/2} L \frac{\dot{L}^2}{4\pi} - \beta_1 Q^{1/2} L \,. \tag{3.19}$$

These yield the conservation law for the topological domain wall

$$\frac{1}{8\pi} L^2 \dot{L}^2 + L^2 = L_0^2 \tag{3.20}$$

while for the Q-wall

$$\frac{1}{8\pi} L \dot{L}^2 + L = L_0 \,, \tag{3.21}$$

where L_0 is the initial value of L. These are easily integrable and yield the time for contraction to be

$$T_{\text{top.-wall}} = \frac{L_0}{2\sqrt{2\pi}} \tag{3.22}$$

and

$$T_{Q\text{-wall}} = \sqrt{\frac{\pi}{2}}\frac{L_0}{4} \tag{3.23}$$

for the topological domain wall and for the Q-wall, respectively. Again we see that the time for contraction is about the same in the two cases.

If the Q-wall spontaneously beads up into a locally planar array of N smaller Q-balls of charge q, we have

$$Q = Nq \tag{3.24}$$

and

$$E = N\beta_3 q^{3/4} = \beta_3 Q^{3/4} N^{1/4} \,. \tag{3.25}$$

Then energy considerations dictate

$$\beta_3 N^{1/4} Q^{3/4} < \beta_1 Q^{1/2} L \,, \tag{3.26}$$

i.e.

$$\left(\frac{\beta_3}{\beta_1}\right)^4 Q_1^{1/2} < \left(\frac{L^2}{N}\right)^{1/2} \tag{3.27}$$

which says the size of each small Q-ball must be greater than the width of the Q-wall.

4. Stability

The general rate of decay is governed by the stability analysis of the fluctuations about the Q-wall or Q-string configuration. The total lagrangian is expanded to second order in the fluctuations about the soliton configuration. The first order terms vanish because the soliton is a solution of the full three-dimensional equations of motion. The second order truncation yields a generalized normal mode problem.

Real frequencies lead to oscillatory behaviour while imaginary frequencies yield exponential growth or decay. In the latter case, the lifetime is taken to the inverse of the magnitude of the imaginary frequency, which is the time scale for the instability leading to beading. We will analyze the problem of the stability of a Q-wall for a specific potential which leads to an analytically accessible problem.

4.1 Analytical Q-walls

As we have seen in section 2.3, the expression for a Q-ball in 1 dimension can be brought to quadrature. For many choices of the potential the resulting integral can be computed analytically [7]. The stability analysis of the corresponding Q-wall is related to the elegant theory of exactly solvable quantum mechanical systems, supersymmetric quantum mechanics and shape invariant potentials [8]. Consider the potential

$$V(\varphi) = \omega^2 \left(\frac{5}{2}\varphi^2 - 2\frac{\varphi^3}{\phi_0} \right) . \tag{4.1}$$

Then

$$-V(\varphi) + \frac{1}{2}\omega^2\varphi^2 = 0 \tag{4.2}$$

implies

$$\varphi^2 = 0\,, \qquad \text{or} \qquad \varphi = \phi_0\,. \tag{4.3}$$

The non-trivial zero is fixed at ϕ_0 and the Q-ball solution will start from this value for φ at its centre. The equation of motion is

$$-\varphi_0''(x) - \omega^2\varphi_0(x) + V'\left(\varphi_0(x)\right) = 0 \tag{4.4}$$

which becomes

$$-\varphi_0''(x) + \omega^2 \left(4\varphi_0(x) - 6\frac{\varphi_0^2(x)}{\phi_0} \right) = 0\,. \tag{4.5}$$

The quadrature (2.19) is integrable in this case, with solution

$$\varphi_0(x) = \phi_0 \operatorname{sech}^2(\omega x)\,. \tag{4.6}$$

It is remarkable that the stability problem around such a Q-wall is related to the corresponding supersymmetric quantum mechanical system, as we will now show.

4.2 Second-order lagrangian

The stability analysis proceeds with the second-order expansion of the total lagrangian about the Q-wall configuration. We will consider an infinite planar Q-wall, which then gives that (4.6) is an exact solution of the full three-dimensional equations. This will be an arbitrarily close approximation for a large but finite closed Q-wall surface. The second-order lagrangian for $\phi(\vec{x},t) = e^{i\omega t}\varphi(\vec{x},t) = e^{i\omega t}(\varphi_0(x) + \varphi_1(\vec{x},t) + i\varphi_2(\vec{x},t))$ is

$$\mathcal{L}^{[2]} = \int d^3x \frac{1}{2}\left(\partial_t\varphi_a - \omega\epsilon_{ab}\varphi_b\right)\left(\partial_t\varphi_a - \omega\epsilon_{ab}\varphi_b\right) + \\ + \frac{1}{2}\partial_i\varphi_a\partial_i\varphi_a - \frac{1}{2}V''\left(\varphi_0\right)\varphi_1^2 - \frac{1}{2}\frac{V'\left(\varphi_0\right)}{\varphi_0}\varphi_2^2\,, \tag{4.7}$$

where $a, b = 1, 2$. Arbitrary variations φ_1 and φ_2 are permitted from the point of view of charge conservation. All spatial Fourier components in the orthogonal directions will correspond to displacement of charge along the wall in a periodic fashion. Charge conservation constraints will only apply to the zero frequency Fourier component, that is constant along the wall, and here the variations will have to be constrained. But this component corresponds to fluctuations that are effectively one dimensional, depending only on the coordinate normal to the Q-wall configuration. We know that the Q-wall is by definition stable under such fluctuations, hence we will not be concerned with these fluctuations. The equations of motion for φ_a are

$$\begin{aligned} \ddot{\varphi}_1 - 2\omega\dot{\varphi}_2 - \omega^2\varphi_1 - \nabla^2\varphi_1 + V''\left(\varphi_0\right)\varphi_1 &= 0\,, \\ \ddot{\varphi}_2 + 2\omega\dot{\varphi}_1 - \omega^2\varphi_2 - \nabla^2\varphi_2 + \frac{V'\left(\varphi_0\right)}{\varphi_0}\varphi_2 &= 0\,. \end{aligned} \tag{4.8}$$

Replacing $\varphi_a(\vec{x},t) \to e^{-i(\Omega t + k_1 y + k_2 z)}\varphi_a(x)$ and writing the resulting equations in matrix form yields

$$\begin{pmatrix} -\frac{d^2}{dx^2} + V''(\varphi_0) - \omega^2 & i2\omega\Omega \\ -i2\omega\Omega & -\frac{d^2}{dx^2} + \frac{V'(\varphi_0)}{\varphi_0} - \omega^2 \end{pmatrix}\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} = \left(\Omega^2 - |\vec{k}|^2\right)\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} \tag{4.9}$$

which is a transcendental equation for Ω.

The operator on the LHS plays the role of a hamiltonian whose eigenvalues determine if a mode is oscillatory or exponential. The hamiltonian is a hermitean operator for real Ω, hence its spectrum is real. It is easy to see that all non-negative eigenvalues of the hamiltonian at $\Omega = 0$ will give rise to oscillatory modes of the fluctuation spectrum. Only negative modes at $\Omega = 0$, for small enough values of $|\vec{k}|$ can yield exponential modes, but even these become oscillatory modes for large enough $|\vec{k}|$. The non-standard aspect of this stability problem is that the hamiltonian also directly depends upon the fluctuation frequency Ω. We must find the spectral curves $E(\Omega)$ as functions of Ω and then look for solutions of the equation $E(\Omega) = \Omega^2 - |\vec{k}|^2$. The RHS is an upward opening parabola for real Ω with intercept at $-|\vec{k}|^2$. The spectral curve of any eigenvalue of the hamiltonian that is non negative at $\Omega = 0$ will intersect with the parabola at two distinct points. This is because for large enough Ω the spectral curves are simply asymptotic to the linear functions $E(\Omega) = \pm 2\omega\Omega$. It is only the eigenvalues of the hamiltonian that are negative at $\Omega = 0$ which can yield exponentially decaying solutions. We will see that there is exactly one such mode for this hamiltonian with the choice of the scalar potential taken in (4.1).

4.3 Spectrum of the hamiltonian and supersymmetric quantum mechanics

The hamiltonian for $\Omega = 0$ is given by

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_+ & 0 \\ 0 & \mathcal{H}_- \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} + V''(\varphi_0) - \omega^2 & 0 \\ 0 & -\frac{d^2}{dx^2} + \frac{V'(\varphi_0)}{\varphi_0} - \omega^2 \end{pmatrix} . \tag{4.10}$$

There is an obvious exact zero energy level,

$$\psi^- = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_0(x) \end{pmatrix} \tag{4.11}$$

since φ_0 satisfies eq. (4.5). Remarkably the mode

$$\psi^+ = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi_0'(x) \\ 0 \end{pmatrix} \tag{4.12}$$

is also an exact zero mode, essentially the corresponding eigenvalue equation is the derivative of the equation of motion for $\varphi_0(x)$ given by eq. (4.5). The notation will become clear below. This is, of course, not just a coincidence. The two sub-hamiltonians which comprise $\mathcal{H}$ are actually supersymmetric partners and their spectra are paired except for the lowest energy mode. Indeed since (4.12) is the derivative of $\varphi_0(x)$ it has a node, hence it cannot be the lowest energy eigenvalue of $\mathcal{H}_+$, and there has to be a negative energy mode.

4.3.1 Stability equations and supersymmetric quantum mechanics [9]

Consider the class of supersymmetric quantum mechanics models determined by the super potential

$$\mathcal{W} = n\omega \tanh(\omega x) . \tag{4.13}$$

Then the corresponding supersymmetric partner hamiltonians are given by

$$\mathcal{H}_\pm = \mathcal{W}^2 \mp \mathcal{W}' = n^2\omega^2 - n(n \pm n)\omega^2 \operatorname{sech}^2 \omega x . \tag{4.14}$$

The two hamiltonians $\mathcal{H}_\pm$ are given in terms of

$$\mathcal{H}_+ = A^\dagger A , \qquad \mathcal{H}_- = AA^\dagger , \tag{4.15}$$

where

$$A = \frac{d}{dx} + \mathcal{W} , \qquad A^\dagger = -\frac{d}{dx} + \mathcal{W} \tag{4.16}$$

which gives

$$\begin{aligned} \mathcal{H}_+ &= -\frac{d^2}{dx^2} + n^2\omega^2 - n(n+1)\omega^2 \operatorname{sech}^2 \omega x , \\ \mathcal{H}_- &= -\frac{d^2}{dx^2} + n^2\omega^2 - n(n-1)\omega^2 \operatorname{sech}^2 \omega x . \end{aligned} \tag{4.17}$$

The ground state is the wave function annihilated by A

$$A\varphi_{-1} = 0 \tag{4.18}$$

which integrates easily to

$$\varphi_{-1}(x) = \varphi_{-1}(0)\,\mathrm{sech}^n\,\omega x\,. \tag{4.19}$$

Then, if $\mathcal{H}_+\psi_+ = E_+\psi_+$ then $\mathcal{H}_-(A\psi_+) = E_+(A\psi_+)$ which shows that all energy levels are (supersymmetrically) paired except for the level annihilated by A. Equivalently if $\mathcal{H}_-\psi_- = E_-\psi_-$ the $\mathcal{H}_+(A^\dagger\psi_-) = E_-(A^\dagger\psi_-)$. This pairing is equivalent to the former except that there is no state annihilated by $A^\dagger$. To find all the levels of $\mathcal{H}_\pm$ we proceed by changing the value of $n \to n-1$. This process continues until we reach $n=1$ and the hamiltonian $\mathcal{H}_-|_{n=1} = -(d^2/dx^2) + \omega^2$ for which we know the spectrum to be free, plane waves. The ground states of all intermediate hamiltonians are given by

$$\psi \sim \mathrm{sech}^m\,\omega x\,, \qquad m = 1, 2, \ldots, n\,. \tag{4.20}$$

Then applying the appropriate strings of $A_m^\dagger$ (using an obvious notation) generates all of the bound states of $\mathcal{H}_\pm$ as shown in detail in [8]. The interesting point is that we can easily find the potential $V(\varphi_0)$ that is necessary to generate this system of supersymmetric hamiltonians for the stability problem of the Q-wall. Indeed, we impose that

$$\begin{aligned} \frac{V'(\varphi_0)}{\varphi_0} &= \omega^2 n(n-1)\,\mathrm{sech}^2\,\omega x + B\,, \\ V''(\varphi_0) &= \omega^2 n(n+1)\,\mathrm{sech}^2\,\omega x + C\,, \end{aligned} \tag{4.21}$$

where B, C are constants to be adjusted. This yields

$$\frac{1}{n-1}\left(\frac{V'(\varphi_0)}{\varphi_0} - B\right) = \omega^2 n\,\mathrm{sech}^2\,\omega x = \frac{1}{n+1}\left(V''(\varphi_0) - C\right). \tag{4.22}$$

This equation is equivalent to

$$\begin{aligned} -\left(\frac{n+1}{n-1}\right)\varphi_0^{-\left(\frac{n+1}{n-1}\right)-1}V'(\varphi_0) + \varphi_0^{-\left(\frac{n+1}{n-1}\right)}V''(\varphi_0) + \\ + \left(\frac{B(n+1) - C(n-1)}{n-1}\right)\varphi_0^{-\left(\frac{n+1}{n-1}\right)} = 0 \end{aligned} \tag{4.23}$$

after multiplying by an integrating factor. Then this equation is easily integrated to

$$V(\varphi_0) = \frac{D\varphi_0^{\left(\frac{n+1}{n-1}\right)+1}}{\left(\frac{n+1}{n-1}\right)+1} + \left(\frac{B(n+1) - C(n-1)}{4}\right)\varphi_0^2 + G \tag{4.24}$$

for some integration constants D, G. Now imposing consistency with eqs. (4.21) yields

$$\frac{V'(\varphi_0)}{\varphi_0} = D\varphi_0^{\left(\frac{2}{n-1}\right)} + \left(\frac{B(n+1) - C(n-1)}{2}\right) = \omega^2 n(n-1)\,\mathrm{sech}^2\,\omega x + B\,. \quad (4.25)$$

Taking $D = \omega^2 n(n-1)$ yields

$$\varphi_0 = \mathrm{sech}^{n-1}\,\omega x\,, \qquad \text{and} \qquad \frac{B(n+1) - C(n-1)}{2} = B\,. \quad (4.26)$$

On the other hand from (4.21)

$$\begin{aligned} V''(\varphi_0) &= D\left(\frac{n+1}{n-1}\right)\varphi_0^{-\left(\frac{n+1}{n-1}\right)-1} + \left(\frac{B(n+1) - C(n-1)}{2}\right) \\ &= \omega^2 n(n+1)\,\mathrm{sech}^2\,\omega x + C \end{aligned} \quad (4.27)$$

hence taking $D = \omega^2 n(n-1)$ as before is consistent since

$$D\left(\frac{n+1}{n-1}\right) = \omega^2 n(n+1) \quad (4.28)$$

as required. The constants however must satisfy (along with the equation from before)

$$\begin{aligned} \frac{B(n+1) - C(n-1)}{2} &= B\,, \\ \frac{B(n+1) - C(n-1)}{2} &= C\,. \end{aligned} \quad (4.29)$$

These are two homogeneous linear equations for B, C which only have the trivial solution except if the system is dependent. It is indeed, and we find the solution is simply $B = C$. We see that the potential is a polynomial for only the cases $n = 2, 3$.

4.4 Stability analysis for analytical Q-walls

The system studied in section 4.1 uses

$$V'\left(\varphi_0(x)\right) = \left(-6\omega^2\frac{\varphi_0(x)}{\phi_0} + 5\omega^2\right)\varphi_0(x) \quad (4.30)$$

hence replacing $\varphi_0(x) = \phi_0\,\mathrm{sech}^2\,\omega x$ yields

$$\frac{V'(\varphi_0(x))}{\varphi_0(x)} = \left(5\omega^2 - 6\omega^2\frac{\varphi_0(x)}{\phi_0}\right) = 5\omega^2 - 6\omega^2\,\mathrm{sech}^2\,\omega x \quad (4.31)$$

while

$$V''(\varphi_0(x)) = \left(5\omega^2 - 12\omega^2\frac{\varphi_0(x)}{\phi_0}\right) = 5\omega^2 - 12\omega^2\mathrm{sech}^2\omega x\,. \quad (4.32)$$

Thus the system corresponds to the case $n = 3$ above with $B = C = 5\omega^2$. The

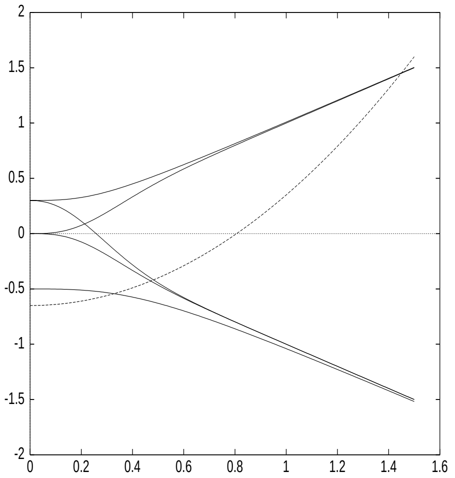

Figure 1: Graphical depiction of transcendental equation for Ω (eq. (4.33), with an appropriate numerical choice for ω); $k^2 > .5$. Solid line represents discrete eigenvalues of the LHS of this equation; continuous eigenvalues are expected to have similar behavior. Broken line represents the RHS. Points of intersection give the (real) values of Ω for the corresponding modes; no intersection would indicate no real solution, indicating an unstable mode of the Q-wall. No such instabilities exist in this case.

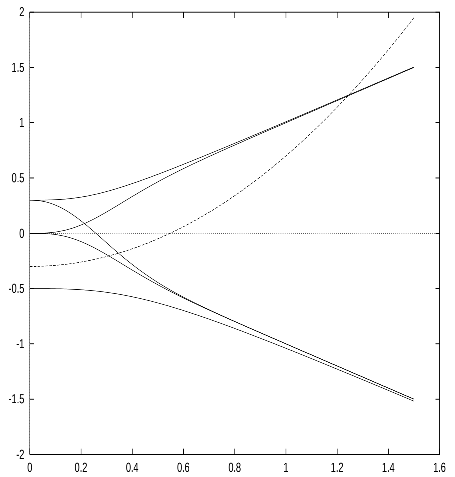

Figure 2: Graphical depiction of transcendental equation for Ω (eq. (4.33), with an appropriate numerical choice for ω); $k^2 < .5$. No solution exists for lowest eigenvalue of the LHS, indicating an unstable mode.

corresponding stability equation is

$$\begin{pmatrix} -\frac{d^2}{dx^2} + 4\omega^2 - 12\omega^2 \operatorname{sech}^2 \omega x & i2\omega\Omega \\ -i2\omega\Omega & -\frac{d^2}{dx^2} + 4\omega^2 - 6\omega^2 \operatorname{sech}^2 \omega x \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \\ = \left(\Omega^2 - |\vec{k}|^2\right) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{4.33}$$

The spectrum of this hamiltonian at $\Omega = 0$ corresponds to a single negative energy bound state with $E = -5\omega^2$

$$\psi^+_{-5\omega^2} = \sqrt{\frac{15\omega}{16}} \begin{pmatrix} \operatorname{sech}^3 \omega x \\ 0 \end{pmatrix} \tag{4.34}$$

two zero energy bound states with $E = 0$

$$\psi^+_0 = \sqrt{\frac{15\omega}{4}} \begin{pmatrix} \operatorname{sech}^2 \omega x \tanh \omega x \\ 0 \end{pmatrix}, \qquad \psi^-_0 = \sqrt{\frac{3\omega}{4}} \begin{pmatrix} 0 \\ \operatorname{sech}^2 \omega x \end{pmatrix} \tag{4.35}$$

and two positive energy bound states with $E = 3\omega^2$

$$\begin{aligned} \psi^+_{3\omega^2} &= \sqrt{24\omega} \begin{pmatrix} 12 \operatorname{sech} \omega x - 15 \operatorname{sech}^3 \omega x \\ 0 \end{pmatrix}, \\ \psi^-_{3\omega^2} &= \sqrt{\frac{3\omega}{2}} \begin{pmatrix} 0 \\ \operatorname{sech} \omega x \tanh \omega x \end{pmatrix} \end{aligned} \tag{4.36}$$

and a continuum starting at $E > 4\omega^2$. All these modes are found in a straightforward manner using the general ideas schematized in section 4.3.1, for more details see [8].

The eigenvalue problem (4.33) has solutions that are best expressed graphically. The spectral curve $E(\Omega)$ of each eigenvalue of the hamiltonian on the LHS of (4.33) must intersect the parabola $\Omega^2 - |\vec{k}|^2$. The eigenvalue problem for this hamiltonian can be written algebraically as

$$(H_0 + H_1\sigma_3 + 2\omega\Omega\sigma_2)\,\psi = E(\Omega)\psi\,, \tag{4.37}$$

where σ_i are the usual Pauli matrices. The spectrum is doubly degenerate for each non-negative mode at $\Omega = 0$ due to the underlying supersymmetry. Clearly

$$(H_0 + H_1\sigma_3 - 2\omega\Omega\sigma_2)\,\sigma_3\psi = E(\Omega)\sigma_3\psi\,; \tag{4.38}$$

hence the spectrum is even in Ω. However we can without loss of generality restrict $\Omega > 0$ because φ_i are in principle real: reversing the sign of $\Omega \to -\Omega$ does not generate an independent set of solutions.

We see clearly that the non-negative modes satisfy $E(\Omega) > \Omega^2 - |\vec{k}|^2$ for $\Omega \approx 0$, but as $\Omega \to \infty$ these spectral curves asymptote to $E(\Omega) \to \pm 2\omega\Omega$, the spectrum of the operator

$$2\omega\Omega\sigma_2\,. \tag{4.39}$$

These are linear functions of Ω which for sufficiently large Ω always satisfy $\Omega^2 - |\vec{k}|^2 > 2\omega\Omega$. Hence by continuity each spectral curve must intersect at least once with the parabola, giving rise to a real solution for Ω, and hence an oscillatory mode in the fluctuation spectrum. We speculate that this occurs exactly once for the spectral curve of each non-negative eigenvalue.

Only the negative mode needs to be treated differently. Here, at $\Omega = 0$ and for $|\vec{k}|^2$ small enough, the negative mode is already below the parabola and hence there is generically no real solution for Ω. We cannot compute the solution analytically, but a perturbative approach is fruitful. We can compute the spectral curve to second order in perturbation theory (the first order change vanishes). We get

$$E(\Omega) = -5\omega^2 - 4\Omega^2\omega^2 \sum_{n=1}^{\infty} \frac{|\langle \psi^+_{-5\omega^2}|\sigma_2|\psi_n\rangle|^2}{E_n + 5\omega^2} + \cdots = -5\omega^2 - \Omega^2 B^2 + o\left(\Omega^4\right), \quad (4.40)$$

where B^2 is a calculable positive real number that is independent of ω. It is not illuminating to compute B^2, it is a number of order 1. Then the equation to be satisfied is

$$-5\omega^2 - \Omega^2 B^2 = \Omega^2 - |\vec{k}|^2 \quad (4.41)$$

that is

$$\Omega^2 = \frac{-5\omega^2 + |\vec{k}|^2}{1 + B^2}. \quad (4.42)$$

For $|\vec{k}|^2 < 5\omega^2$ this gives $\Omega^2 < 0$, i.e. the solution for Ω is imaginary and the corresponding mode is unstable. Indeed

$$\Omega = i\omega\sqrt{\frac{5 - \left(|\vec{k}|^2/\omega^2\right)}{1 + B^2}}\,; \quad (4.43)$$

hence as $|\vec{k}|^2 \to 5\omega^2$ from above, we get $\Omega \to 0$. The lifetime is defined as $T = 1/|\Omega|$ hence for small values of $|\vec{k}|$ we get $T \sim (1/\omega)$. But as $|\vec{k}|^2 \to 5\omega^2$ we get $T \to \infty$. Thus the very long wavelength spatial Fourier modes decay with lifetime of the order of $1/\omega$ while as the wavelength $1/|\vec{k}|$ approaches $1/5\omega$, which is of the order of the thickness of the Q-wall, the lifetime becomes arbitrarily long, the modes become quasi stable. For larger $|\vec{k}|$ (shorter wavelength) the modes are of course strictly stable. Analysis of related, exactly solvable simple models such as $H(\Omega)\psi = (k^2 + \alpha\sigma_3 + \Omega\sigma_2)\psi = \Omega^2\psi$ yields the same kind of behavior, generically. Hence we can conclude that the fluctuation spectrum corresponds to stable oscillatory modes except for one degree of freedom. The corresponding mode is unstable only if the wavelength of the spatial Fourier mode is larger than the width of the Q-wall. Furthermore, the lifetime of the unstable mode is larger than $1/\omega \sim Q_1^{(1/2)} = (Q/L^2)^{(1/2)}$, which clearly can be taken to be arbitrarily large. Hence we conclude that Q-wall, or equally well a Q-string can be arbitrarily long-lived.

5. Discussion and conclusion

In this paper we have studied lower-dimensional Q-balls and extended objects based on them in models with a scalar field potential with a flat direction for large field values. The energy of a Q-ball of charge Q_d in d space dimensions varies as $Q_d^{(d/d+1)}$.

JHEP08(2001)003

Thus, as three-dimensional Q-balls become more and more energetically advantageous with increasing charge, the same is true (only more so) in two or in one dimension.

The extended configurations, called Q-strings and Q-walls, also have interesting energetics. A string of fixed charge and of varying length L, for example, has energy $E \sim L^{1/3}$, indicating a rapidly decreasing tension as the length is increased. The time scale of oscillations turns out to be of the same order as that for a topological string.

These strings and walls have a different instability not shared by topological strings: namely, density fluctuations along the string or wall can form, causing a sort of beading: the extended object would evolve into a one- or two-dimensional array of smaller Q-balls.

This instability was examined in detail for Q-walls, where supersymmetric quantum mechanics enables a rather detailed analysis. Only modes corresponding to long-wavelength fluctuations are unstable, and the time scale of growth of these fluctuations can be rather large (compared to the characteristic time scale of the Q-wall).

It is not clear whether or why such objects would form in the early Universe. (Unlike conventional cosmic strings, there is no topological impetus for their formation.) But if they did form, they could be long-lived, and could help contribute to structure formation in the Universe. Such a possibility merits further investigation.

Acknowledgments

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